Spectrum for Heavy Quankonia and Mixture of the Relevant Wave Functions within the Framework of Bethe-Salpeter Equation

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Abstract

Considering the fact that some excited states of the heavy quarkonia (charmonium and bottomonium) still missing in experimental observations and potential applications of the relevant wave functions of the bound states, we re-analyze the spectrum and the relevant wave functions of the heavy quarkonia within the framework of Bethe-Salpeter (B.S.) equation with a proper QCD-inspired kernel. Such a kernel for the heavy quarkonia, relating to potential of non-relativistic quark model, is instantaneous, so we call the corresponding B.S. equation as BS-In equation throughout the paper. Particularly, a new way to solve the B.S. equation, which is different from the traditional ones, is proposed here, and with it not only the known spectrum for the heavy quarkonia is re-generated, but also an important issue is brought in, i.e., the obtained solutions of the equation 'automatically' include the 'fine', 'hyperfine' splittings and the wave function mixture, such as $S-D$ wave mixing in $J^{PC}=1^{--}$ states, $P-F$ wave mixing in $J^{PC}=2^{++}$ states for charmonium and bottomonium etc. It is pointed out that the best place to test the wave mixture probably is at Z-factory (e^+e^- collider running at Z-boson pole with extremely high luminosity).

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I. INTRODUCTION

Spectroscopy, including the spectrum and the corresponding wave functions, is a very interesting topic for heavy quarkonia in particle physics. The spectrum and the corresponding wave functions for the binding systems can be tested experimentally and via study of the spectroscopy one may have insight of the heavy quarkonia and understand QCD , which is the nature of the binding, further as well. In the literature, there are various approaches to the spectroscopy of the heavy quarkonia: charmonium and bottomonium [1–7], and to solve the Bethe-Salpeter (B.S.) equation is one of them [4–7]. Since recently we have realized a new method to solve the B.S. equation for the heavy quarkonia, so in this paper, we would like to try the method i.e. to apply this method to re-analyzing the spectroscopy of the heavy quarkoia: charmonium and bottomonium under B.S. equation approach.

First of all, how to determine the B.S. kernel is crucial for B.S. equation approach to a bound state problem. It is known that if one adopts the QCD-inspired Bethe-Salpeter (B.S.) equation [8] approach to the problems of hadronic bound states, then the relevant B.S. kernel for a double heavy quark-antiquark system, such as charmonium and bottomonium, is instantaneous approximately i.e. the B.S. equation is essentially an instantaneous one (a BS-In equation). It is also known that BS-In equation can further precisely relate to the Schödinger equation in potential model (PM) by means of the Salpeter approximate method [9]. Therefore, one may use the relation to the potential model (PM) and help oneself to determine the kernel of BS-In equation precisely. Whereas starting with the BS-In equation whose kernel is fixed in terms of QCD consideration and the relation to PM, one can extend some relativistic nature of the problem more than what PM can consider, and stand on more solid theoretical ground for the B.S. equation approach, hence, we start the study of spectroscopy for heavy quarkia with such a BS-In equation. Moreover, we apply the new realized method to solving the BS-In equation. People later on will see an important issue from the new method is that besides the 'fine and hyperfine' splitting being involved, the wave mixtures in the wave functions, such as $S-D$ wave mixing in $J^{PC}=1^{--}$ (J: total angle momentum; P: parity; C: charge parity) states and $P-F$ wave mixing in $J^{PC}=2^{++}$ states etc, are determined precisely, although the mixtures, in fact, are rooted in the kernel under the present framework of BS-In equation.

The new proposed method can be outlined as that, firstly we analyze the bound states according to their total angular momentum J, parity P and charge conjugation C, such as the states $0^{-+}(^1S_0)$, $1^{--}(^3S_1$ or $^3D_1)$, $0^{++}(^3P_0)$, $1^{++}(^3P_1)$, $2^{++}(^3P_2$ or $^3F_2)$, and $1^{+-}(^1P_1)$ etc;

secondly we write down the most general formulation for the B.S. wave functions respectively, and then input the formulated wave functions into the BS-In equation and turn the equation into a set of proper coupled equations for the components which appear in the formulation; finally we solve the coupled equation numerically, and obtain the mass spectra and wave functions for $(c\bar{c})$ and $(b\bar{b})$ binding systems. For convenience, we call the coupled equations as BS-CoEqs later on.

This paper is organized as following, additional to the Introduction section I, in section II we introduce the relativistic Bethe-Salpeter equation and BS-In equation. In section III we start with the generalized formulation for relativistic wave functions with definite quantum numbers to derive the relevant BS-CoEqs for low total angle momentum states individually. Finally, we show the numerical results although we do not present the detail to solve the equation numerically, and we also explain and briefly discuss the obtained solutions of the BS-CoEqs in section IV.

II. INSTANTANEOUS BETHE-SALPETER EQUATION

Firstly let us outline the reduction of the B.S. equation which is similar to the way of Salpeter [9] if B.S. kernel is instantaneous, and introduce necessary notations. The readers, who are interested in the details, can also find them in Ref. [10–12].

The Bethe-Salpeter (B.S.) equation for mesons is read as:

$$(\not{p}_1 - m_1)\chi(q)(\not{p}_2 + m_2) = i\int \frac{d^4k}{(2\pi)^4} V(P,k,q)\chi(k)\ , \tag{1}$$

where $\chi(q)$ is the B.S. wave function, $V(P,k,q)$ is the interaction kernel between the quark and antiquark, and p_1, p_2 are the momenta of the quark 1 and anti-quark 2. Quark mass is m_1, antiquark mass is m_2, and here we consider heavy quarkonia: charmonium and bottomonium so we have $m_1 = m_2$. The total momentum P and the relative momentum q are defined as:

$$p_1 = \alpha_1 P + q, \ \ \alpha_1 = \frac{m_1}{m_1+m_2}\ ,$$
$$p_2 = \alpha_2 P - q, \ \ \alpha_2 = \frac{m_2}{m_1+m_2}\ ,$$

and $\alpha_1 = \alpha_2 = \frac{1}{2}$ for charmonium and bottomonium.

We divide the relative momentum q into two parts, $q_\parallel$ and $q_\perp$,

$$q^\mu = q^\mu_\parallel + q^\mu_\perp\ ,$$

$$q_{\parallel}^{\mu} \equiv (P\cdot q/M^2)P^{\mu}\ , \quad q_{\perp}^{\mu} \equiv q^{\mu} - q_{\parallel}^{\mu}\ .$$

Correspondingly, we may have two Lorentz invariant variables:

$$q_{_P} = \tfrac{(P\cdot q)}{M}\ , \qquad q_{_T} = \sqrt{q_{_P}^2 - q^2} = \sqrt{-q_{\perp}^2}\ .$$

When $\vec{P}$= 0, they turn to the usual component q_0 and $|\vec{q}|$ respectively.

If the kernel $V(P,k,q)$ takes the simple form:

$$V(P,k,q) \Rightarrow V(k_{\perp}, q_{\perp})$$

namely the B.S. equation is 'instantaneous', for convenience, we would like to introduce the notations $\varphi_p(q_{\perp}^{\mu})$ and $\eta(q_{\perp}^{\mu})$ so the 'instantaneous (three dimensional) objects' will accordingly read as follows:

$$\varphi_{_P}(q_{\perp}^{\mu}) \equiv i\int \frac{dq_{_P}}{2\pi}\chi(q_{\parallel}^{\mu}, q_{\perp}^{\mu})\ , \tag{2}$$

$$\eta(q_{\perp}^{\mu}) \equiv \int \frac{dk_{\perp}}{(2\pi)^3}V(k_{\perp}, q_{\perp})\varphi_{_P}(k_{\perp}^{\mu})\ . \tag{3}$$

The B.S. equation now is rewritten as:

$$\chi(q_{\parallel}, q_{\perp}) = S_1(p_1)\eta(q_{\perp})S_2(p_2)\ . \tag{4}$$

Generally the propagators of the two constituents can be decomposed as:

$$S_i(p_i) = \frac{\Lambda_{iP}^{+}(q_{\perp})}{J(i)q_{_P} + \alpha_i M - \omega_i + i\epsilon} + \frac{\Lambda_{iP}^{-}(q_{\perp})}{J(i)q_{_P} + \alpha_i M + \omega_i - i\epsilon}\ , \tag{5}$$

with

$$\omega_i = \sqrt{m_i^2 + q_{_T}^2}\ , \quad \Lambda_{iP}^{\pm}(q_{\perp}) = \frac{1}{2\omega_i}\left[\frac{\not{P}}{M}\omega_i \pm J(i)(m_i + \not{q}_{\perp})\right]\ , \tag{6}$$

where $i = 1, 2$ for quark and anti-quark respectively, and $J(i) = (-1)^{i+1}$, and $\Lambda_{iP}^{\pm}(q_{\perp})$ satisfy the relations:

$$\Lambda_{iP}^{+}(q_{\perp}) + \Lambda_{iP}^{-}(q_{\perp}) = \frac{\not{P}}{M}\ , \ \ \Lambda_{iP}^{\pm}(q_{\perp})\frac{\not{P}}{M}\Lambda_{iP}^{\pm}(q_{\perp}) = \Lambda_{iP}^{\pm}(q_{\perp})\ , \ \ \Lambda_{iP}^{\pm}(q_{\perp})\frac{\not{P}}{M}\Lambda_{iP}^{\mp}(q_{\perp}) = 0\ . \tag{7}$$

Hence sometimes $\Lambda_{iP}^{\pm}(q_{\perp})$ are called as 'project operators', although they need to be sandwiched with the operator $\frac{\not{P}}{M}$ when 'projecting' as Eq(7).

Introducing the notations $\varphi_{_P}^{\pm\pm}(q_{\perp})$ to note the projected wave functions as:

$$\varphi_{_P}^{\pm\pm}(q_{\perp}) \equiv \Lambda_{1P}^{\pm}(q_{\perp})\frac{\not{P}}{M}\varphi_{_P}(q_{\perp}^{\mu})\frac{\not{P}}{M}\Lambda_{2P}^{\pm}(q_{\perp})\ , \tag{8}$$

and we indeed have

$$\varphi_{_P}(q_\perp^\mu) = \varphi_{_P}^{++}(q_\perp^\mu) + \varphi_{_P}^{+-}(q_\perp^\mu) + \varphi_{_P}^{-+}(q_\perp^\mu) + \varphi_{_P}^{--}(q_\perp^\mu)$$

With contour integration over q_p on both sides of Eq.(4), we obtain:

$$\varphi_{_P}(q_\perp) = \frac{\Lambda_{1P}^+(q_\perp)\eta(q_\perp)\Lambda_{2P}^+(q_\perp)}{(M-2\omega_1)} - \frac{\Lambda_{1P}^-(q_\perp)\eta(q_\perp)\Lambda_{2P}^-(q_\perp)}{(M+2\omega_1)} \ ,$$

and the equation becomes four independent equations:

$$\begin{aligned}
(M-2\omega_1)\varphi_{_P}^{++}(q_\perp) &= \Lambda_{1P}^+(q_\perp)\eta(q_\perp)\Lambda_{2P}^+(q_\perp) \ , \\
(M+2\omega_1)\varphi_{_P}^{--}(q_\perp) &= -\Lambda_{1P}^-(q_\perp)\eta(q_\perp)\Lambda_{2P}^-(q_\perp) \ , \\
\varphi_{_P}^{+-}(q_\perp) = \varphi_{_P}^{-+}(q_\perp) &= 0 \ .
\end{aligned} \tag{9}$$

where we have $\omega_1 = \omega_2$ for the equal mass system. In fact the four equations is of an 'eigenvalue problem' about the eigenvalue M. Note that in the Ref.[9] the way for solving the BS-In equation is not exactly equivalent to the four equations Eq.(9). Details about examining the equivalence may be found in Ref.[10]. Alternately here we exactly start with the four equations to solve the BS-In equation.

The normalization condition for B.S. wave function is:

$$\int \frac{q_{_T}^2 dq_{_T}}{2\pi^2} Tr\left[\overline{\varphi}^{++}\frac{\not P}{M}\varphi^{++}\frac{\not P}{M} - \overline{\varphi}^{--}\frac{\not P}{M}\varphi^{--}\frac{\not P}{M}\right] = 2P_0 \ . \tag{10}$$

Now let us return to the problem for the heavy quarkonia $(c\bar{c})$ and $(b\bar{b})$. To fix the kernel for the heavy quark and heavy anti-quark, on one hand, we should let the kernel being QCD-inspired and on the other hand, we should relate the kernel to the Cornell potential accordingly. Thus the kernel in space-time looks like as a linear scalar interaction (the confinement one in QCD nonperturbative nature) plus a vector interaction (single gluon exchange in Coulomb gauge):

$$V(r) = V_s(r) + \gamma_0 \otimes \gamma^0 V_v(r) = \lambda r + V_0 - \gamma_0 \otimes \gamma^0 \frac{4}{3}\frac{\alpha_s}{r} \ , \tag{11}$$

where λ is the string constant, $\alpha_s(r)$ is the running coupling constant of QCD. Usually, in order to fit the data of heavy quarkonia, a constant V_0 is often added to the scalar confining potential and takes different values for the bound states with different quantum numbers respectively[1].

[1] One will see later on in this paper that the value of V_0 is determined by fitting the data for the ground states with the corresponding quantum numbers.

To avoid the infrared divergence in the Coulomb-like one and to correspond the fact that the confined linear interaction should be also suppressed at large distance phenomenologically, so it will be better to re-formulate the kernel as follows:

$$V_s(r) = \frac{\lambda}{\alpha}(1 - e^{-\alpha' r}) + V_0 \ ,$$
$$V_v(r) = -\frac{4}{3}\frac{\alpha_s}{r}e^{-\alpha r} \ . \tag{12}$$

To decrease the parameters which are needed to fix by fitting data, we assume $\alpha' = \alpha$ approximately[2]. It is easy to show that when $\alpha r \ll 1$, the potential approximately becomes linear. Now the B.S. kernel in momentum space and in the rest frame of the bound state is read as:

$$V(\vec{q}) = V_s(\vec{q}) + \gamma_0 \otimes \gamma^0 V_v(\vec{q}) \ ,$$
$$V_s(\vec{q}) = -(\frac{\lambda}{\alpha} + V_0)\delta^3(\vec{q}) + \frac{\lambda}{\pi^2}\frac{1}{(\vec{q}^2 + \alpha^2)^2} \ ,$$
$$V_v(\vec{q}) = -\frac{2}{3\pi^2}\frac{\alpha_s(\vec{q})}{(\vec{q}^2 + \alpha^2)} \ ,$$
$$\alpha_s(\vec{q}) = \frac{12\pi}{33 - 2N_f}\frac{1}{\log(a + \frac{\vec{q}^2}{\Lambda^2_{QCD}})} \tag{13}$$

where $N_f = 3$ for $(c\bar{c})$ system, $N_f = 4$ for $(b\bar{b})$ system; the constants λ, α, a, V_0 and Λ_{QCD} are the parameters which characterize the kernel (potential).

III. GENERAL FORMULATION FOR THE B.S. WAVE FUNCTIONS AND THE COUPLED EQUATIONS

In fact in this section, we show the new realized method to solve a BS-In equation, but specifically apply to the concerned heavy quarkonium problem.

Firstly, according to the total angle momentum (J), parity (P) and charge conjugation (C) of the concerned bound state, we write down the most general formulation for each of the relativistic B.S. wave functions, and then we put it into Eq.(9) to derive out the coupled equation for the components appearing in the formulation, BS-CoEq. In the below subsections, we do the derivation for the low-laying states: $J^{PC} = 0^{-+}, 1^{--}, 1^{+-}, 0^{++}, 1^{++}, 2^{++}, etc$ in turn precisely.

[2] In fact, at final step (numerical solving BS-CoEq) we find that the results are not very sensitive to the assumption when α and α' vary in reasonable region.

A. $J^{PC} = 0^{-+}$, 1^{+-}, 0^{++} and 1^{++} wave functions and BS-CoEqs for relevant components

Since the bound states with the quantum numbers $J^{PC} = 0^{-+}, 1^{+-}, 0^{++}$ and 1^{++} are similar, so in this subsection we derive the equations for them in turn.

I. *The bound states with quantum numbers* $J^{PC} = 0^{-+}$, which in non-relativistic framework are ${}^1S_0$ states mainly.

The general formulation of the In-BS wave function Eq.(2) for the states $J^{PC} = 0^{-+}$ is [11, 12]:

$$\varphi_{P,0^{-+}}(q_\perp^\mu) = \varphi_{P,0^{-+}}(q_{_T}) = \left[\not{P} f_1(q_{_T}) + M f_2(q_{_T}) + \frac{\not{P}\not{q}_\perp}{M} f_3(q_{_T}) \right] \gamma_5 \ , \tag{14}$$

where M is the mass of the bound state (the corresponding meson) and $q_\perp^\mu = q^\mu - \frac{(P\cdot q)}{M^2} P^\mu$ is the four dimensional vector. In the center mass system $q_\perp^\mu = (0, \vec{q})$, $q_{_T} = \mid \vec{q} \mid$. Later on we abbreviate $q_{_T}$ as q if it does not make any confusion.

Now let us derive the coupled equations (BS-CoEqs) from Eq.(9). To put Eq.(14) into the last two equations of Eq.(9)

$$\varphi^{+-}_{P,0^{-+}}(q) = \varphi^{-+}_{P,0^{-+}}(q) = 0$$

and by taking various traces for γ-matrices on both sides of the equations, we obtain the independent constraints on the components for the wave function:

$$f_3(q) = -\frac{f_1(q) M}{m_1} \ , \tag{15}$$

so we can apply the obtained constraints Eq.(15) to Eq.(14) and rewrite the relativistic wave function of state 0^{-+} as:

$$\begin{aligned} \varphi_{P,0^{-+}}(q) &= \left[\not{P} f_1(q) + M f_2(q) + \not{q}_\perp \frac{\not{P}}{m_1} f_1(q) \right] \gamma_5 \\ &= \left[(1 + \frac{\not{q}_\perp}{m_1}) \not{P} f_1(q) + M f_2(q) \right] \gamma_5 \ . \end{aligned} \tag{16}$$

From the above formulation of the wave function one can see clearly that besides the 'great component', which is proportional to either $M\gamma_5$ or $\not{P}\gamma_5$, there is also a 'small component', which is proportional to $\frac{\not{q}_\perp}{m_1}\not{P}\gamma_5$, linear in $q_\perp$ (P-wave nature) and suppressed by $\frac{1}{m_1}$.

Put the wave function Eq.(16) into the first two equations of Eq.(9) and by taking various traces for γ-matrices to both sides of the equations, we obtain the independent coupled

integral equations (BS-CoEqs):

$$(M-2\omega_1)\left[f_1(q)+f_2(q)\frac{m_1}{\omega_1}\right]=-\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{\omega_1^2}$$
$$\times\left\{(V_s-V_v)\left[f_1(k)m_1^2+f_2(k)m_1\omega_1\right]-(V_s+V_v)f_1(k)(\vec{q}\cdot\vec{k})\right\}\ ,$$
$$(M+2\omega_1)\left[f_1(q)-f_2(q)\frac{m_1}{\omega_1}\right]=\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{\omega_1^2}$$
$$\times\left\{(V_s-V_v)\left[f_1(k)m_1^2-f_2(k)m_1\omega_1\right]-(V_s+V_v)f_1(k)(\vec{q}\cdot\vec{k})\right\}\ , \tag{17}$$

here $k\equiv|\vec{k}|$, $\omega_1=\sqrt{m_1^2+q_T^2}$. Now we are prepare ready to solve the BS-CoEqs Eq.(17), as an eigenvalue problem, for f_1 and f_2 numerically, specially in center mass system, and we may obtain the required results (the spectrum for $J^{PC}=0^{-+}$ states and the B.S. wave functions accordingly) finally.

Now accordingly the normalization condition is read as

$$\int\frac{d^3q}{(2\pi)^3}4f_1(q)f_2(q)M^2\left\{\frac{\omega_1}{m_1}+\frac{m_1}{\omega_1}+\frac{q^2}{\omega_1 m_1}\right\}=2M\ . \tag{18}$$

II. *The bound states with quantum numbers* $J^{PC}=1^{+-}$ which in non-relativistic framework are $^1P_1$ states mainly:

As that for the states $J^P=1^{+-}$, the general form of the In-BS wave function can be written as [12, 16]:

$$\varphi_{_{P,1^{+-}}}(q)=q_\perp\cdot\epsilon_\perp^\lambda\left[f_1(q)+f_2(q)\frac{\not{P}}{M}+f_3(q)\frac{\not{P}\not{q}_\perp}{M^2}\right]\gamma_5. \tag{19}$$

From the equations

$$\varphi^{+-}_{_{P,1^{+-}}}(q)=\varphi^{-+}_{_{P,1^{+-}}}(q)=0\ , \tag{20}$$

a constraint on the components of the wave function

$$f_3(q)=-\frac{f_2(q)M}{m_1}$$

is obtained.

With the constraint, the wave function now turns into:

$$\begin{aligned}\varphi_{_{P,1^{+-}}}(q)\ &=\ q_\perp\cdot\epsilon_\perp^\lambda\left[f_1(q)+f_2(q)\frac{\not{P}}{M}-f_2(q)\frac{\not{P}\not{q}_\perp}{m_1M}\right]\gamma_5\\ &=\ q_\perp\cdot\epsilon_\perp^\lambda\left[f_1(q)+f_2(q)\Big(1+\frac{\not{q}_\perp}{m_1}\Big)\frac{\not{P}}{M}\right]\gamma_5\,,\end{aligned} \tag{21}$$

here the factor $(q_\perp \cdot \epsilon_\perp^\lambda)$ indicates the wave function is of P-wave nature mainly; whereas in Eq.(21) the 'small component' term $(q_\perp \cdot \epsilon_\perp^\lambda)\frac{\not{q}_\perp}{m_1}\frac{\not{P}}{M}\gamma_5$ contains high order wave.

Now the normalization condition for the $^1P_1$ wave function is read as:

$$\int \frac{d^3\vec{q}}{(2\pi)^3}\frac{4f_1 f_2 \omega_1 q^2}{3m_1} = M. \tag{22}$$

In terms of the same derivation as that for $J^{PC} = 0^{-+}$ states, we obtain the coupled equations (BS-CoEqs) for the components f_1 and f_2:

$$\begin{aligned}
(M-2\omega_1)\left[f_1(q) + f_2(q)\frac{\omega_1}{m_1}\right] &= \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{(\vec{k}\cdot\vec{q})}{\omega_1 m_1 q^2} \\
\times \left\{(V_s+V_v)f_2(k)(\vec{k}\cdot\vec{q}) - (V_s-V_v)\left[f_1(k)\omega_1 m_1 + f_2(k)m_1^2\right]\right\}&, \\
(M+2\omega_1)\left[f_1(q) - f_2(q)\frac{\omega_1}{m_1}\right] &= -\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{(\vec{k}\cdot\vec{q})}{\omega_1 m_1 q^2} \\
\times \left\{-(V_s+V_v)f_2(k)(\vec{k}\cdot\vec{q}) - (V_s-V_v)\left[f_1(k)m_1\omega_1 - f_2(k)m_1^2\right]\right\}&.
\end{aligned} \tag{23}$$

Therefore, we are prepare ready to solve the coupled equations BS-CoEqs, as an eigenvalue problem, for f_1 and f_2 numerically, specially in center mass system, and we may obtain the required results (the spectrum for $J^{PC} = 1^{+-}$ states and the B.S. wave function accordingly) finally.

III. *The bound states* $J^{PC} = 0^{++}$ *and* 1^{++} which in non-relativistic framework essentially are $^3P_0$ and $^3P_1$ states respectively:

Since the bound states $J^{PC} = 0^{++}$ and 1^{++} are very similar, thus here we treat them simultaneously. For the states $J^P = 0^{++}$, the general form of the In-BS wave functions can be written as [12, 16]:

$$\varphi_{_{P,0^{++}}}(q) = f_1(q)\not{q}_\perp + f_2(q)\frac{\not{P}\not{q}_\perp}{M} + f_3(q)M\ . \tag{24}$$

With the equations

$$\varphi^{+-}_{_{P,0^{++}}}(q) = \varphi^{-+}_{_{P,0^{++}}}(q) = 0\ , \tag{25}$$

we obtain the constraints:

$$f_3(q) = -\frac{f_1(q)q^2}{Mm_1}\ .$$

Then the wave function:

$$\begin{aligned}
\varphi_{_{P,0^{++}}}(q) &= f_1(q)\not{q}_\perp + f_2(q)\frac{\not{q}_\perp\not{P}}{M} - \frac{f_1(q)\vec{q}^{\,2}}{m_1} \\
&= \not{q}_\perp\Big[(1-\frac{\not{q}_\perp}{m_1})f_1(q) + f_2(q)\frac{\not{P}}{M}\Big]\ .
\end{aligned} \tag{26}$$

Here the factor $\not{q}_\perp = (\vec{q}\cdot\vec{\gamma})$ in CMS is contained in the wave functions, that means the the wave function is of P-wave nature. Whereas the term which contains $\vec{q}^{\,2} = (\not{q}_\perp)(\not{q}_\perp)$ in Eq.(26) is suppressed by the factor $\frac{1}{M}$.

In terms of the same way as that for $J^{PC} = 0^{-+}$ states, with the first two equations of Eq.(9) we obtain the coupled equations for the $J^{PC} = 0^{++}$ states:

$$\begin{aligned}(M-2\omega_1)\left[f_1(q)+f_2(q)\frac{m_1}{\omega_1}\right] &= \int\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{\omega_1^2}\times\Big\{(V_s+V_v)\left[-f_1(k)q^2\right]k^2\\ &+(m_1)(V_s-V_v)\left[f_1(k)m_1+f_2(k)\omega_1\right](\vec{k}\cdot\vec{q})\Big\}\,;\\ (M+2\omega_1)\left[f_1(q)-f_2(q)\frac{m_1}{\omega_1}\right] &= \int\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{\omega_1^2}\times\Big\{(V_s+V_v)\left[f_1(k)q^2\right]k^2\\ &-(m_1)(V_s-V_v)\left[f_1(k)m_1-f_2(k)\omega_1\right](\vec{k}\cdot\vec{q})\Big\}\,. \end{aligned}\tag{27}$$

The normalization condition for the wave function is read:

$$\int\frac{d^3\vec{q}}{(2\pi)^3}\frac{4f_1f_2\omega_1q^2}{m_1} = M\,.\tag{28}$$

Whereas for the $J^{PC} = 1^{++}$ states, the general form for the wave function can be written as [12, 16]:

$$\begin{aligned}\varphi_{_{P,1^{++}}}(q) = i\varepsilon_{\mu\nu\alpha\beta}P^\nu q_\perp^\alpha\epsilon^\beta\Big[&f_1(q)M\gamma^\mu+f_2(q)\not{P}\gamma^\mu\\ &+if_3(q)\varepsilon^{\mu\rho\sigma\delta}q_{\perp\rho}P_\sigma\gamma_\delta\gamma_5/M\Big]/M^2\,.\end{aligned}\tag{29}$$

From the he equations

$$\varphi^{+-}_{_{P,1^{++}}}(q) = \varphi^{-+}_{_{P,1^{++}}}(q) = 0\,,\tag{30}$$

we obtain the constraints on the components of the wave function:

$$f_3(q) = \frac{f_2(q)M}{m_1}\,.$$

Then we have:

$$\varphi_{_{P,1^{++}}}(q) = i\varepsilon_{\mu\nu\alpha\beta}P^\nu q_\perp^\alpha\epsilon^\beta\Big[f_1(q)M\gamma^\mu+f_2(q)(\not{P}\gamma^\mu+i\varepsilon^{\mu\rho\sigma\delta}q_{\perp\rho}P_\sigma\gamma_\delta\gamma_5/m_1)\Big]/M^2\,.\tag{31}$$

Here the front factor $\varepsilon_{\mu\nu\alpha\beta}P^\nu q_\perp^\alpha\epsilon^\beta$, being linear in $\vec{q}$, means the wave functions are of P-wave nature.

In terms of the same way as that for $J^{PC} = 0^{-+}$ states, with the first two equations of Eq.(9) we obtain the coupled equations (BS-CoEqs) as follows:

$$(M-2\omega_1)\left[f_1(q)+f_2(q)\frac{\omega_1}{m_1}\right] = \int\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_1m_1\vec{k}^2\vec{q}^2}\times\Big\{-(V_s+V_v)f_2(k)\left[k^2q^2+(\vec{k}\cdot\vec{q})^2\right]$$

$$- 2m_1(V_s - V_v)\left[f_1(k)\omega_1 + f_2(k)m_1\right]k^2(\vec{k}\cdot\vec{q})\Big\}$$

$$(M+2\omega_1)\left[f_1(q) - f_2(q)\frac{\omega_1}{m_1}\right] = -\int\frac{d\vec{k}}{(2\pi)^3}\frac{1}{2\omega_1 m_1 k^2 q^2}\times\Big\{(V_s+V_v)f_2(k)\left[k^2q^2 + (\vec{k}\cdot\vec{q})^2\right]$$
$$- 2m_1(V_s - V_v)\left[f_1(k)\omega_1 - f_2(k)m_1\right]k^2(\vec{k}\cdot\vec{q})\Big\} \tag{32}$$

The normalization condition for the $J^{PC} = 1^{++}$ wave function is read:

$$\int\frac{d^3\vec{q}}{(2\pi)^3}\frac{8f_1f_2\omega_1q^2}{3m_1} = M\,. \tag{33}$$

Now we are ready to solve the coupled equations Eqs.(27, 32) numerically.

B. $J^{PC} = 1^{--}, 2^{++}$ wave functions for In-BS equation and BS-CoEqs for relevant components

As for the states $J^{PC} = 1^{--}, 2^{++}$, they are quite different from the states in the above subsection, because there is $S-D$ wave mixing in the $J^{PC} = 1^{--}$ states and there is $P-F$ wave mixing in the $J^{PC} = 2^{++}$ states.

I. *The bound states* $J^{PC} = 1^{--}$ which in non-relativistic framework are $^3S_1$ and/or $^3D_1$ states mainly:

First of all, we write down the general formulation for the wave functions of In-BS equation with quantum numbers $J^P = 1^{--}$[12, 15]:

$$\varphi^{\lambda}_{P,1^{--}}(q) = q_\perp\cdot\epsilon^\lambda_\perp\left[f_1(q) + \frac{\not{q}_\perp}{M}f_3(q_\perp) + \frac{\not{P}\not{q}_\perp}{M^2}f_4(q)\right] + M\not{\epsilon}^\lambda_\perp f_5(q)$$
$$+\not{\epsilon}^\lambda_\perp\not{P}f_6(q) + \frac{1}{M}(\not{P}\not{\epsilon}^\lambda_\perp\not{q}_\perp - \not{P}q_\perp\cdot\epsilon^\lambda_\perp)f_2(q)\,, \tag{34}$$

where the $\epsilon^\lambda_\perp$ is the polarization vector of the vector meson. From the last two equations of Eq.(9)

$$\varphi^{\lambda,+-}_{P,1^{--}}(q) = \varphi^{\lambda,-+}_{P,1^{--}}(q) = 0\,, \tag{35}$$

we obtain the independent constraints on the components of the wave functions:

$$f_1(q) = \frac{-q^2f_3(q) + M^2f_5(q)}{Mm_1}\,,\qquad f_2(q) = -\frac{f_6(q)M}{m_1}\,.$$

Then with the constraints, there are only four independent components $f_3(q)$, $f_4(q)$, $f_5(q)$ and $f_6(q)$ left in the Eq.(34). Namely

$$\varphi^{\lambda}_{1^{--}}(q_\perp) = q_\perp\cdot\epsilon^\lambda_\perp\Big(\frac{-q^2}{Mm_1} + \frac{\not{q}_\perp}{M}\Big)f_3(q) + q_\perp\cdot\epsilon^\lambda_\perp\frac{\not{P}\not{q}_\perp}{M^2}f_4(q)$$
$$+\Big(M\not{\epsilon}^\lambda_\perp + q_\perp\cdot\epsilon^\lambda_\perp\frac{M}{m_1}\Big)f_5(q) + \Big[\not{\epsilon}^\lambda_\perp\not{P} + \frac{\not{P}(q_\perp\cdot\epsilon^\lambda_\perp)}{m_1} - \frac{(\not{P}\not{\epsilon}^\lambda_\perp\not{q}_\perp)}{m_1}\Big]f_6(q), \tag{36}$$

and from the formulation it is easy to realize that the 'great components' in the wave function, which are proportional to f_5 or f_6 and $\not{\epsilon}_\perp^\lambda$ or $(\not{\epsilon}_\perp^\lambda \not{P})$ are of S-wave nature, whereas the components in the wave function, which are proportional to f_3 or f_4 and $(q_\perp \cdot \epsilon_\perp^\lambda)\not{q}_\perp$ or $(q_\perp \cdot \epsilon_\perp^\lambda)\not{q}_\perp \not{P}$ (double $q_\perp$ being contained) are of D-wave nature (a tensor about $q_\perp$). Therefore, no matter what are the other 'small terms', the wave functions Eq.(36) involve $S-D$ wave mixing properly.

To put Eq.(36) into the first two equations of Eq.(9) and take various traces on both sides of the equations, we obtain four coupled integral equations for the four independent components f_3, f_4, f_5 and f_6 (BS-CoEqs):

$$
\begin{aligned}
&(M-2\omega_1)\left\{\left(f_3(q)\frac{q^2}{M^2}-f_5(q)\right)+\left(f_4(q)\frac{q^2}{M^2}+f_6(q)\right)\frac{m_1}{\omega_1}\right\}\\
&=\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{2}{\omega_1^2}\left\{(V_s+V_v)\left(f_3(k)\frac{k^2}{M^2}-f_5(k)\right)(\vec{k}\cdot\vec{q})\right.\\
&\left.-(V_s-V_v)\left[m_1^2\left(f_3(k)\frac{(\vec{k}\cdot\vec{q})^2}{M^2q^2}-f_5(k)\right)+m_1\omega_1\left(f_4(k)\frac{(\vec{k}\cdot\vec{q})^2}{M^2q^2}+f_6(k)\right)\right]\right\}, \qquad (37)
\end{aligned}
$$

$$
\begin{aligned}
&(M+2\omega_1)\left\{\left(f_3(q)\frac{q^2}{M^2}-f_5(q)\right)-\left(f_4(q)\frac{q^2}{M^2}+f_6(q)\right)\frac{m_1}{\omega_1}\right\}\\
&=-\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{2}{\omega_1^2}\left\{(V_s+V_v)\left[\left(f_3(k)\frac{k^2}{M^2}-f_5(k)\right)\right](\vec{k}\cdot\vec{q})\right.\\
&\left.-(V_s-V_v)\left[m_1^2\left(f_3(k)\frac{(\vec{k}\cdot\vec{q})^2}{M^2q^2}-f_5(k)\right)-m_1\omega_1\left(f_4(k)\frac{(\vec{k}\cdot\vec{q})^2}{M^2q^2}+f_6(k)\right)\right]\right\}, \qquad (38)
\end{aligned}
$$

$$
\begin{aligned}
&(M-2\omega_1)\left\{\left(f_3(q)+f_4(q)\frac{m_1}{\omega_1}\right)\frac{q^2}{M^2}-3\left(f_5(q)-f_6(q)\frac{\omega_1}{m_1}\right)-f_6(q)\frac{q^2}{m_1\omega_1}\right\}\\
&=-\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{\omega_1^2}\left\{(V_s+V_v)\left[-\frac{2\omega_1}{m_1}f_6(k)-f_3(k)\frac{k^2}{M^2}+f_5(k)\right](\vec{k}\cdot\vec{q})\right.\\
&+(V_s-V_v)\left[\omega_1^2\left(f_3(k)\frac{k^2}{M^2}-3f_5(k)\right)+m_1\omega_1\left(f_4(k)\frac{k^2}{M^2}+3f_6(k)\right)\right.\\
&\left.\left.-\left(f_3(k)\frac{(\vec{k}\cdot\vec{q})^2}{M^2}-f_5(k)\vec{q}^2\right)\right]\right\}, \qquad (39)
\end{aligned}
$$

$$
\begin{aligned}
&(M+2\omega_1)\left\{\left[f_3(q)-f_4(q)\frac{m_1}{\omega_1}\right]\frac{q^2}{M^2}-3\left(f_5(q)+f_6(q)\frac{\omega_1}{m_1}\right)+f_6(q)\frac{q^2}{m_1\omega_1}\right\}\\
&=\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{\omega_1^2}\left\{(V_s+V_v)\left[\frac{2\omega_1}{m_1}f_6(k)-f_3(k)\frac{k^2}{M^2}+f_5(k)\right](\vec{k}\cdot\vec{q})\right.
\end{aligned}
$$

$$+(V_s-V_v)\left[\omega_1^2\left(f_3(k)\frac{k^2}{M^2}-3f_5(k)\right)-m_1\omega_1\left(f_4(k)\frac{k^2}{M^2}+3f_6(k)\right)\right.$$
$$\left.\left.-\left(f_3(k)\frac{(\vec{k}\cdot\vec{q})^2}{M^2}-f_5(k)q^2\right)\right]\right\}. \tag{40}$$

Now the normalization condition for the wave functions with the components $f_3(q)$, $f_4(q)$, $f_5(q)$ and $f_6(q)$ is read as follows:

$$\int\frac{d\vec{q}}{(2\pi)^3}\frac{16\omega_1\omega_2}{3}\left\{3f_5f_6\frac{M^2}{2m_1\omega_1}+\frac{q^2}{2m_1\omega_1}\left[f_4f_5-f_3\left(f_4\frac{q^2}{M^2}+f_6\right)\right]\right\}=2M. \tag{41}$$

Thus the results about the mass spectra and the wave functions for the $J^{PC}=1^{--}$ bound states can be obtained by solving the coupled Eqs.(37-40) numerically.

II. *The bound states with quantum numbers* $J^{PC}=2^{++}$ which in non-relativistic framework are ${}^3P_2$ and/or ${}^3F_2$ states mainly:

The general form of the wave function for $J^{PC}=2^{++}$ states can be written down as:

$$\varphi^{\lambda}_{P,2^{++}}(q)=\varepsilon^{\lambda}_{\mu\nu}q_\perp^\nu\left\{q_\perp^\mu\Big[f_1(q)+\frac{\not{q}_\perp}{M}f_3(q)+\frac{\not{P}\not{q}_\perp}{M^2}f_4(q)\Big]\right.$$
$$\left.+\gamma^\mu\Big[Mf_5(q)+\not{P}f_6(q)\Big]+\frac{i}{M}f_2(q)\epsilon^{\mu\alpha\beta\gamma}P_\alpha q_{\perp\beta}\gamma_\gamma\gamma_5\right\}, \tag{42}$$

where the $\varepsilon^{\lambda}_{\mu\nu}$ (symmetric in μ and ν) is the tensor polarization of the meson. From the last two equations of Eq.(9):

$$\varphi^{\lambda,+-}_{P,2^{++}}(q)=\varphi^{\lambda,-+}_{P,2^{++}}(q)=0\,, \tag{43}$$

we obtain the constraints on the components of the wave function:

$$f_1(q)=\frac{-q^2f_3(q)+M^2f_5(q)}{Mm_1}\,,\qquad f_2(q)=\frac{f_6(q)M}{m_1}\,.$$

Put the constraints into Eq.(42), then only four independent components $f_3(q)$ $f_4(q)$, $f_5(q)$ and $f_6(q)$ are left:

$$\varphi^{\lambda}_{P,2^{++}}(q_\perp)=\varepsilon^{\lambda}_{\mu\nu}q_\perp^\nu\left\{q_\perp^\mu\left[\Big(\frac{\not{q}_\perp}{M}-\frac{q^2}{Mm_1}\Big)f_3(q)+\frac{\not{P}\not{q}_\perp}{M^2}f_4(q)\right]\right.$$
$$\left.+(\gamma^\mu+\frac{q_\perp^\mu}{m_1})Mf_5(q)+\gamma^\mu\not{P}f_6(q)+i\frac{f_6(q)}{m_1}\epsilon^{\mu\alpha\beta\gamma}P_\alpha q_{\perp\beta}\gamma_\gamma\gamma_5\right\}. \tag{44}$$

From the formulation Eq.(44), it is easy to see that the terms, which is proportional to f_5 or f_6 and with the factor $(\varepsilon^{\lambda}_{\mu\nu}q_\perp^\nu\gamma^\mu)M$ or $(\varepsilon^{\lambda}_{\mu\nu}q_\perp^\nu\gamma^\mu)\not{P}$, are of P-wave nature (linear in $q_\perp$), and

the terms, which proportional to f_3 or f_4 and $(\varepsilon^{\lambda}_{\mu\nu}q_\perp^\nu q_\perp^\mu)\frac{\not P}{M}$ or $(\varepsilon^{\lambda}_{\mu\nu}q_\perp^\nu q_\perp^\mu)\frac{\not P\not q_\perp}{M^2}$, are of F-wave nature (cubic in $q_\perp$). Therefore, Eq.(44) describes $P-F$ wave mixing properly.

Put Eq.(44) into the first two equations of Eq.(9) and take various traces for γ-matrix on both sides of the equations, we obtain the coupled equations for the four independent components f_3, f_4, f_5 and f_6 as follows:

$$
\begin{aligned}
&(M-2\omega_1)\left\{\left(f_3(q)\frac{q^2}{M^2}-f_5(q)\right)+\left(f_4(q)\frac{q^2}{M^2}+f_6(q)\right)\frac{m_1}{\omega_1}\right\}\\
&=\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_1^2q^4}\left\{-(V_s+V_v)q^2\left(f_3(k)\frac{k^2}{M^2}-f_5(k)\right)\left[k^2q^2-3(\vec{k}\cdot\vec{q})^2\right]\right.\\
&\quad+m_1(V_s-V_v)\left[m_1\left(f_3(k)\frac{k^2q^2-3(\vec{k}\cdot\vec{q})^2}{M^2}+2f_5(\vec{k})q^2\right)\right.\\
&\quad\left.\left.+\omega_1\left(f_4(k)\frac{k^2q^2-3(\vec{k}\cdot\vec{q})^2}{M^2}-2f_6(k)q^2\right)\right](\vec{k}\cdot\vec{q})\right\}\ ;
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
&(M+2\omega_1)\left\{\left(f_3(q)\frac{q^2}{M^2}-f_5(q)\right)-\left(f_4(q)\frac{q^2}{M^2}+f_6(q)\right)\frac{m_1}{\omega_1}\right\}\\
&=-\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_1^2q^4}\left\{-(V_s+V_v)q^2\left(f_3(k)\frac{k^2}{M^2}-f_5(k)\right)\left[k^2q^2-3(\vec{k}\cdot\vec{q})^2\right]\right.\\
&\quad+m_1(V_s-V_v)\left[m_1\left(f_3(k)\frac{k^2q^2-3(\vec{k}\cdot\vec{q})^2}{M^2}+2f_5(k)q^2\right)\right.\\
&\quad\left.\left.-\omega_1\left(f_4(k)\frac{k^2q^2-3(\vec{k}\cdot\vec{q})^2}{M^2}-2f_6(k)q^2\right)\right](\vec{k}\cdot\vec{q})\right\}\ ;
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
&(M-2\omega_1)\left\{-f_5(q)m_1+f_6(q)\omega_1\right\}=\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{\omega_1q^2}\\
&\times\left\{-\frac{1}{2}(V_s+V_v)f_6(k)\left[k^2q^2-3(\vec{k}\cdot\vec{q})^2\right]+(V_s-V_v)m_1\left[(f_5(k)\omega_1-f_6(k)m_1)\right.\right.\\
&\quad\left.\left.-\left(f_3(k)\omega_1+f_4(k)m_1\right)\frac{k^2}{M^2}+\left(f_3(k)\omega_1+f_4(k)m_1\right)\frac{(\vec{k}\cdot\vec{q})^2}{M^2q^2}\right](\vec{k}\cdot\vec{q})\right\}\ ;
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
&(M+2\omega_1)\left\{-f_5(q)m_1-f_6(q)\omega_1\right\}=-\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{\omega_1q^2}\\
&\times\left\{\frac{1}{2}(V_s+V_v)f_6(k)\left[k^2q^2-3(\vec{k}\cdot\vec{q})^2\right]-(V_s-V_v)m_1\left[(f_5(k)\omega_1+f_6(k)m_1)\right.\right.\\
&\quad\left.\left.-\left(f_3(k)\omega_1-f_4(k)m_1\right)\frac{k^2}{M^2}+\left(f_3(k)\omega_1-f_4(k)m_1\right)\frac{(\vec{k}\cdot\vec{q})^2}{M^2q^2}\right](\vec{k}\cdot\vec{q})\right\}
\end{aligned}
\tag{48}
$$

TABLE I: Parameter of V_0 in unit of MeV.

	$c\bar{c}$	$b\bar{b}$
$\mathbf{n}J^{PC} = \mathbf{n}\ 0^{-+}(^1S_0)$	-0.314	-0.240
$\mathbf{n}J^{PC} = \mathbf{n}\ 1^{--}(^3S_1)$	-0.176	-0.166
$\mathbf{n}J^{PC} = \mathbf{n}\ 0^{++}(^3P_0)$	-0.282	-0.174
$\mathbf{n}J^{PC} = \mathbf{n}\ 1^{++}(^3P_1)$	-0.162	-0.141
$\mathbf{n}J^{PC} = \mathbf{n}\ 2^{++}(^3P_2)$	-0.110	-0.121
$\mathbf{n}J^{PC} = \mathbf{n}\ 1^{+-}(^1P_1)$	-0.144	-0.135

Now the normalization condition is read as:

$$\int \frac{d^3\vec{q}}{(2\pi)^3}\frac{8\omega_1 q^2}{15m_1}\left\{5f_5f_6M^2 + 2f_4f_5q^2 - 2q^2f_3\left(f_4\frac{q^2}{M^2} + f_6\right)\right\} = 2M. \tag{49}$$

By solving the coupled equations Eqs.(45-48) numerically, we obtain the mass spectra and relevant B.S. wave functions for the $J^{PC} = 2^{++}$ bound states.

In fact, with the way described here, one may derive the BS-In equation for the other possible J^{PC} states into their BS-CoEqs according one's wish.

IV. NUMERICAL RESULTS AND DISCUSSIONS

In this section we solve the equations (BS-CoEqs) numerically and discuss the obtained results.

Since the coupled integration equations are quite complicated, so we solve them numerically only, and additionally with certain approximation such as that a cut on the up-bound of the integrations in the equations has been made.

To solve the equations, we also need to fix the parameters appearing in the kernel Eq.(13) although the kernel is based on QCD inspirer and the Cornell potential for non-relativistic heavy quark model as reference. Usually, the parameters are fixed by fitting the best experimental data. Since now quite a lot of data about the charmonium and bottomnium with quantum data $J^{PC} = 1^{--}$ are available and quite precise, so the most parameters are fixed by the data. Since V_0 in the kernel originates from QCD non-perturbative effects, its value

TABLE II: Mass spectra of $(c\bar{c})$ and $(b\bar{b})$ systems with quantum numbers $J^{PC} = 0^{-+}$, 1^{--}. Here $^{(2S+1)}L_J$ denotes the dominant component in the state respectively. 'Ex' means the experimental results from PDG [17] (and the data for η_b come from reference [18].

n $J^{PC}(^{(2S+1)}L_J)$	Th$(c\bar{c})$	Ex$(c\bar{c})$	Th$(b\bar{b})$	Ex$(b\bar{b})$
1 $0^{-+}(^1S_0)$	2980.3(input)	2980.3	9390.2(input)	9388.9
2 $0^{-+}(^1S_0)$	3576.4	3637	9950.0	
3 $0^{-+}(^1S_0)$	3948.8		10311.4	
1 $1^{--}(^3S_1)$	3096.9(input)	3096.916	9460.5(input)	9460.30
2 $1^{--}(^3S_1)$	3688.1	3686.09	10023.1	10023.26
3 $1^{--}(^3D_1)$	3778.9	3772.92	10129.5	
4 $1^{--}(^3S_1)$	4056.8	4039	10368.9	10355.2
5 $1^{--}(^3D_1)$	4110.7	4153	10434.7	
6 $1^{--}(^3S_1)$	4329.4	4421	10635.8	10579.4
7 $1^{--}(^3S_1)$	4545.9		10852.1	10865

is to account the states with various J^{PC}, so we fix it by fitting the mass of the ground states. Thus the parameter V_0 vary with J^{PC}.

By fitting data, the values of the parameters for all of the states are those as follows:

$$a = e = 2.7183\ , \quad \alpha = 0.06\ \text{GeV}, \quad \lambda = 0.21\ \text{GeV}^2,$$
$$\text{and} \quad m_c = 1.62\ \text{GeV}, \quad m_b = 4.96\ \text{GeV}\,. \tag{50}$$

Since the running coupling constant is used, so we also need to fix Λ_{QCD}. There are three active flavors for $(c\bar{c})$ system, i.e. $N_f = 3$, accordingly we adopt $\Lambda_{QCD} = 0.27$ GeV and the coupling constant at the scale of charm quark mass, $\alpha_s(m_c) = 0.38$. There are four active flavors for $(b\bar{b})$ system, i.e. $N_f = 4$, so $\Lambda_{QCD} = 0.20$ GeV, and the coupling constant $\alpha_s(m_b) = 0.23$. By the fitting ground state data mainly, the fixed value of V_0 for various J^{PC} states is listed in TABLE I.

The spectrum of charmonium and bottomnium (ground states and excited states) obtained by solving the coupled equations numerically is shown in TABLE II and TABLE III. Note here that since the couple-channel effects for the states above the threshold of

TABLE III: Mass spectra of $(c\bar{c})$ and $(b\bar{b})$ systems with quantum numbers $J^{PC} = 0^{++}$, 1^{++}, 2^{++}, 1^{+-} in unit of MeV. Here $^{(2S+1)}L_J$ denotes the dominant component in the state respectively. 'Ex' means the experimental results from PDG [17].

n $J^{PC}(^{2S+1}L_J)$	Th$(c\bar{c})$	Ex$(c\bar{c})$	Th$(b\bar{b})$	Ex$(b\bar{b})$
1 $0^{++}(^3P_0)$	3414.7(input)	3414.75	9859.0	9859.44
2 $0^{++}(^3P_0)$	3836.8		10240.6	10232.5
3 $0^{++}(^3P_0)$	4140.1		10524.7	
1 $1^{++}(^3P_1)$	3510.3(input)	3510.66	9892.2	9892.78
2 $1^{++}(^3P_1)$	3928.7		10272.7	10255.46
3 $1^{++}(^3P_1)$	4228.8		10556.2	
1 $2^{++}(^3P_2)$	3556.1(input)	3556.20	9914.4	9912.21
2 $2^{++}(^3P_2)$	3972.4		10293.6	10268.65
3 $2^{++}(^3F_2)$	4037.9		10374.4	
4 $2^{++}(^3P_2)$	4271.0		10561.5	
1 $1^{+-}(^1P_1)$	3526.0(input)	3525.93	9900.2	
2 $1^{+-}(^1P_1)$	3943.0		10280.4	
3 $1^{+-}(^1P_1)$	4242.4		10562.0	

'open-charm' or 'open-bottom' respectively have not been taken into account, so the results in TABLE II and TABLE III above the threshold cannot compare with experimental results directly.

From the tables, one can read out the fine and hyperfine splitting precisely which are caused by the kernel with the fixed parameters. Therefore not only the gaps among the excited states and the ground states with fixed J^{PC} but also the fine and hyperfine splitting among the states with different J^{PC} are serious tests of the kernel and the B.S. approach to the heavy quarkonia.

We cannot show all the numerical results of the wave functions which we have obtained here, alternatively, as typical examples, we only show some of the obtained wave functions with different quantum numbers J^{PC} in figures FIGs.1,$\cdots$,8.

As usual cases, from the number of nodes of the wave functions in the figures we can

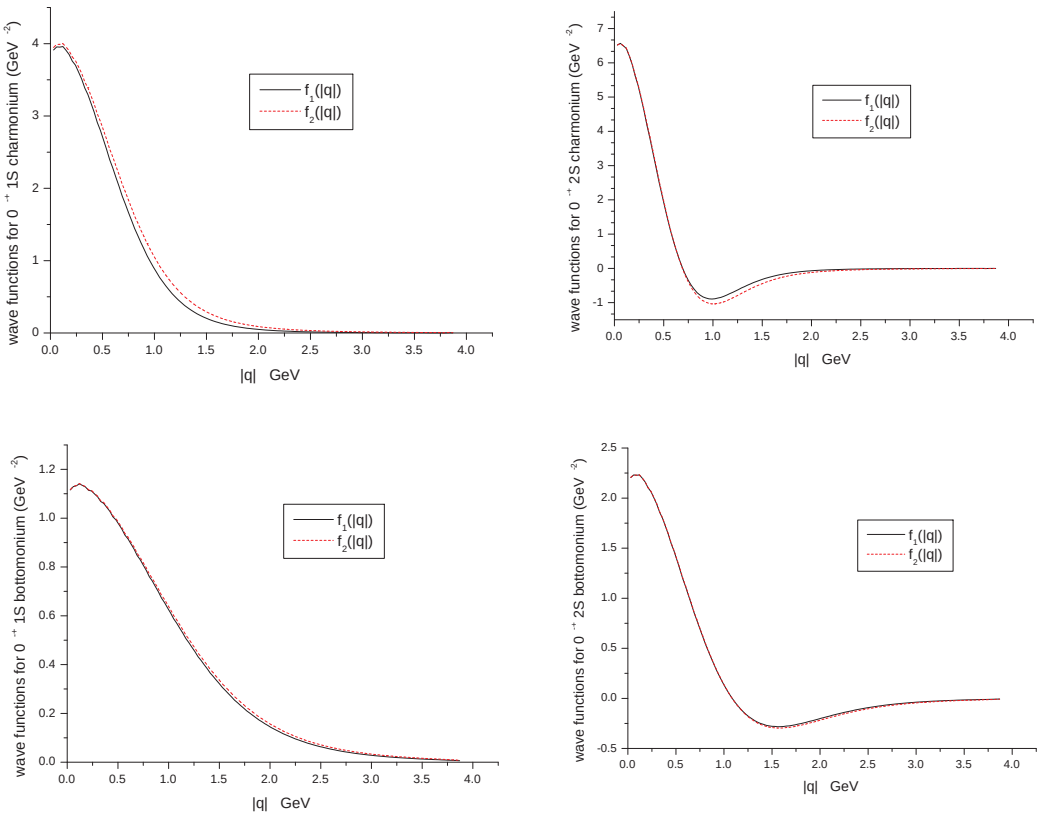

FIG. 1: The solutions for the wave functions of the ground and the first excited state (from left to right) with quantum number $J^{PC} = 0^{-+}$. The wave functions (solutions) of the low-lying states (the ground and the first excited state) with quantum number $J^{PC} = 0^{-+}$. The above two are those for charmonium and the below two are those for bottonium.

realize how high an excited one or the ground each one of the obtained wave functions is.

From FIG.1, we see clearly that for the states $J^{PC} = 0^{-+}$ the solution has the property $f_1 \simeq f_2$, so we may re-write the wave function Eq.(16) as

$$\begin{aligned} \varphi_{P,0^{-+}}(q) &= \left[(1 + \frac{\not{q}_\perp}{m_1})\not{P} f_1(q) + M f_2(q)\right]\gamma_5 \\ &\simeq \phi_{0^{-+}}(q)\left[(1 + \frac{\not{q}_\perp}{m_1})\not{P} + M\right]\gamma_5 \ , \end{aligned} \tag{51}$$

here $\phi_{0^{-+}}(q) \simeq f_1(q) \simeq f_2(q)$, the numerical solution of Eq(17).

From FIGs.2,3,4, we may see that the situations for the states with quantum numbers $J^{PC} = 1^{+-}$, $J^{PC} = 0^{++}$ and $J^{PC} = 1^{++}$ are similar. Indeed the wave functions of the

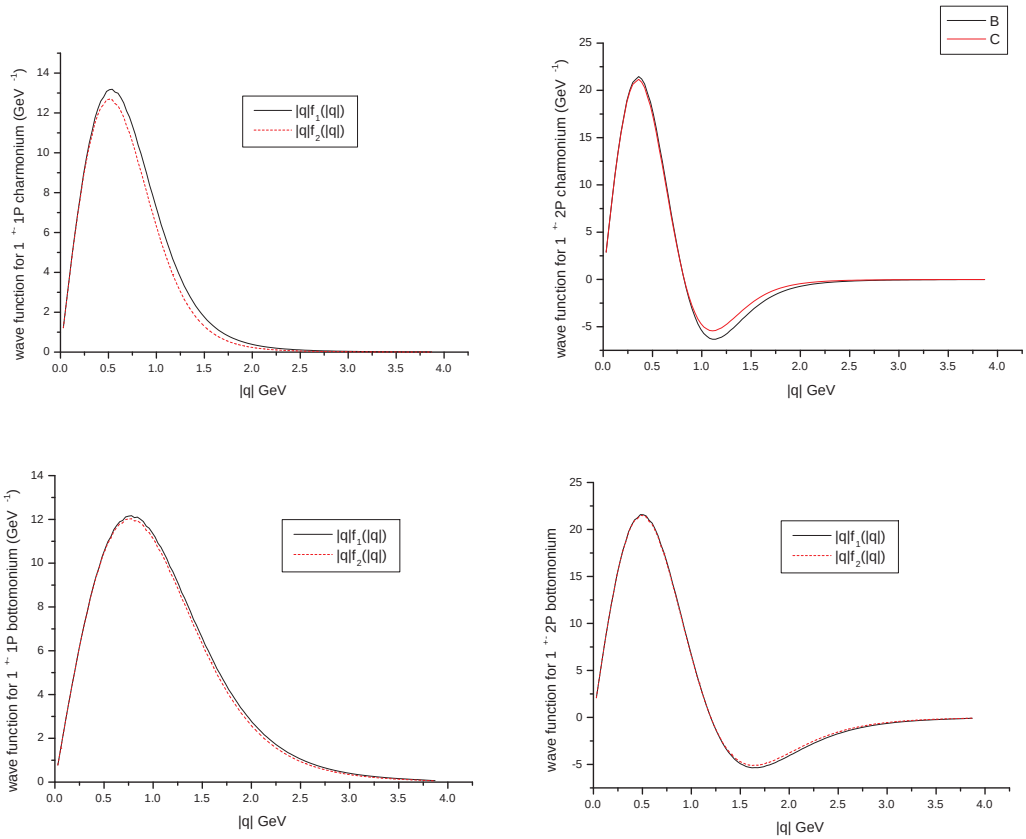

FIG. 2: The solutions for the wave functions of the ground and the first excited state (from left to right) with quantum number $J^{PC} = 1^{+-}$. The wave functions (solutions) of the low-lying states (the ground and the first excited state) with quantum number $J^{PC} = 1^{+-}$. The above two are those for charmonium and the below two are those for bottonium.

ground state and the excited states for $J^{PC} = 1^{+-}$ Eq.(21) become

$$\begin{aligned}\varphi_{P,1^{+-}}(q) &= q_\perp \cdot \epsilon_\perp^\lambda \left[f_1(q) + f_2(q)\Big(1 + \frac{\not{q}_\perp}{m_1}\Big)\frac{\not{P}}{M}\right]\gamma_5 \\ &\simeq \phi_{1^{+-}}(q)(q_\perp \cdot \epsilon_\perp^\lambda)\left[1 + \Big(1 + \frac{\not{q}_\perp}{m_1}\Big)\frac{\not{P}}{M}\right]\gamma_5 \,, \end{aligned} \tag{52}$$

here $\phi_{1^{+-}}(q) \simeq f_1(q) \simeq f_2(q)$, the numerical solution of Eq.(23); the wave function for $J^{PC} = 0^{++}$ Eq.(26) becomes

$$\begin{aligned}\varphi_{P,0^{++}}(q) &= f_1(q)\not{q}_\perp + f_2(q)\frac{\not{q}_\perp \not{P}}{M} - \frac{f_1(q)\vec{q}^2}{m_1} \\ &\simeq \phi_{0^{++}}(q)\Big[1 + \frac{\not{P}}{M} + \frac{\not{q}_\perp}{m_1}\Big]\not{q}_\perp \,, \end{aligned} \tag{53}$$

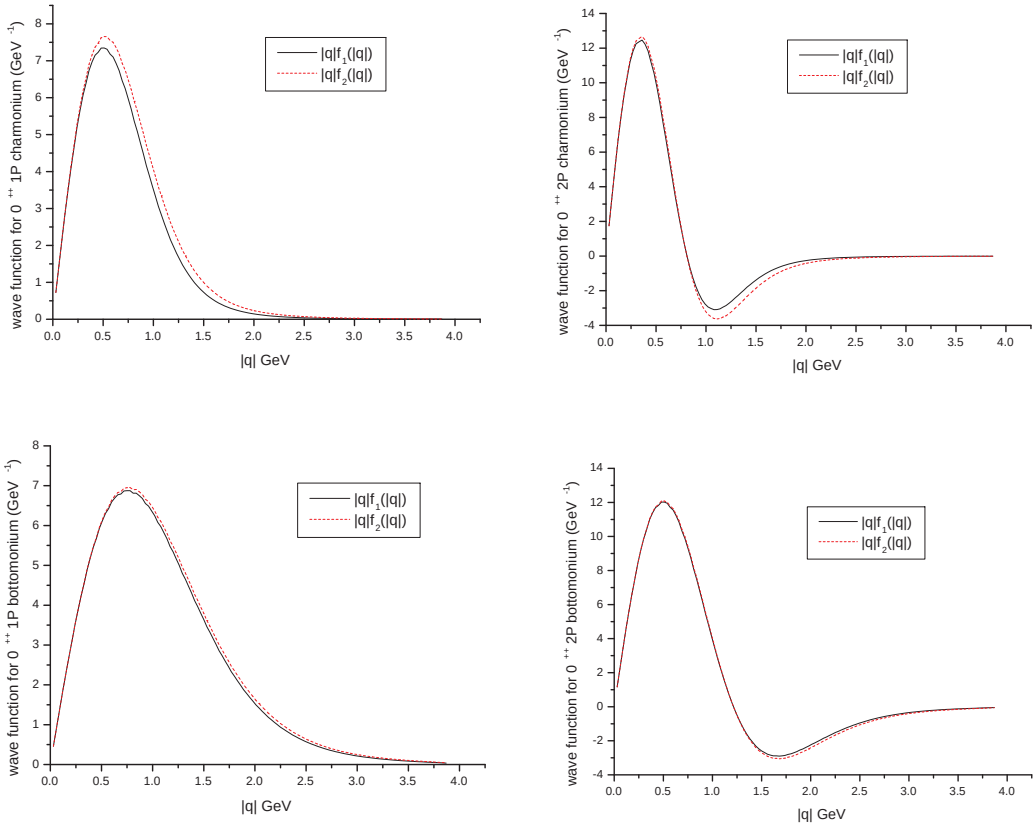

FIG. 3: The solutions for the wave functions of the ground and the first excited state (from left to right) with quantum number $J^{PC} = 0^{++}$. The wave functions (solutions) of the low-lying states (the ground and the first excited state) with quantum number $J^{PC} = 0^{++}$. The above two are those for charmonium and the below two are those for bottonium.

here $\phi_{_{0^{++}}}(q) \simeq f_1(q) \simeq -f_2(q)$, the numerical solution; the wave functions for $J^{PC} = 1^{++}$ Eq.(31) become

$$\begin{aligned}\varphi_{_{P,1^{++}}}(q) &= i\varepsilon_{\mu\nu\alpha\beta}P^\nu q_\perp^\alpha \epsilon^\beta \Big[f_1(q)M\gamma^\mu + f_2(q)\not{P}\gamma^\mu + if_2(q)\varepsilon^{\mu\rho\sigma\delta}q_{\perp\rho}P_\sigma\gamma_\delta\gamma_5/m_1\Big]/M^2 \\ &\simeq i\phi_{_{1^{++}}}(q)\varepsilon_{\mu\nu\alpha\beta}P^\nu q_\perp^\alpha \epsilon^\beta \Big[M\gamma^\mu + \not{P}\gamma^\mu + i\varepsilon^{\mu\rho\sigma\delta}q_{\perp\rho}P_\sigma\gamma_\delta\gamma_5/m_1\Big]/M^2\,, \qquad (54)\end{aligned}$$

here $\phi_{_{1^{++}}}(q) \simeq f_1(q) \simeq -f_2(q)$, the numerical solution of Eq.(32).

Moreover from FIG.5 and FIG.6 one may see the $S-D$ wave mixing for the $J^{PC} = 1^{--}$ states, and from FIG.7 and FIG.8 one may see the $P-F$ wave mixing for the $J^{PC} = 2^{++}$ states clearly. For the $J^{PC} = 1^{--}$ states, from the figures (FIG.5 and FIG.6) we can see that for the first two states (the solutions for the ground one and the first excited one) the S-wave components f_5 and f_6 are dominant, and for the third state (the solution for the second

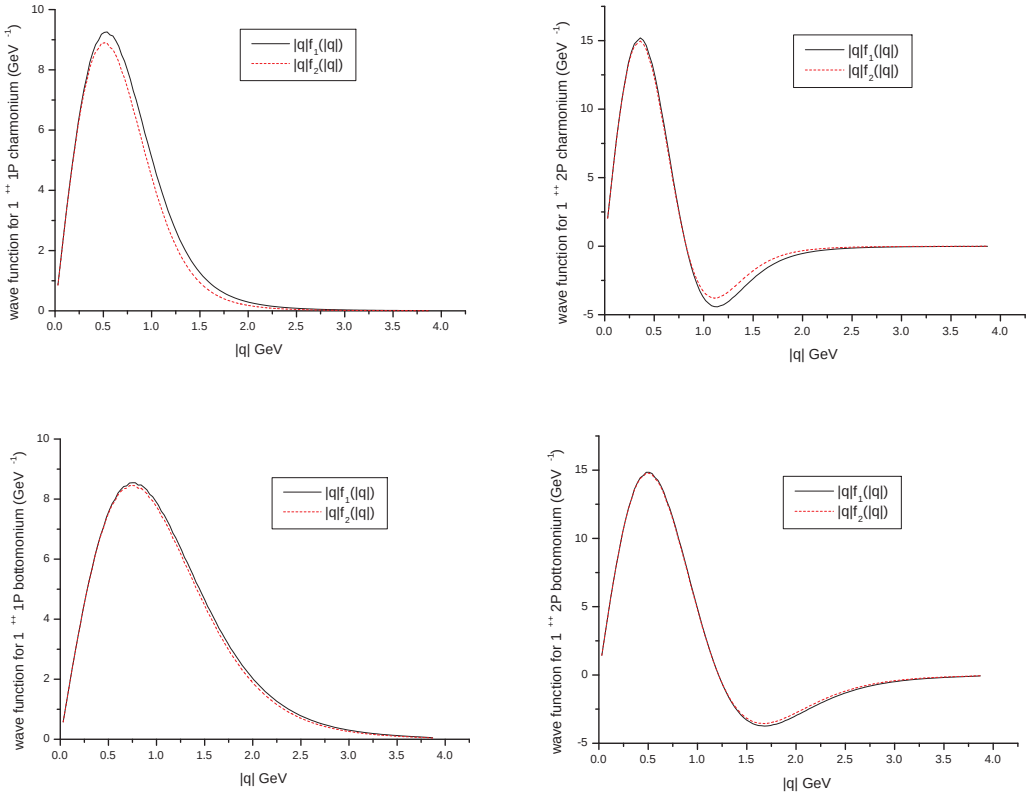

FIG. 4: The solutions for the wave functions of the ground and the first excited state (from left to right) with quantum number $J^{PC} = 1^{++}$. The wave functions (solutions) of the low-lying states (the ground and the first excited state) with quantum number $J^{PC} = 1^{++}$. The above two are those for charmonium and the below two are those for bottonium.

excited one) the D-wave components f_3 and f_4 are dominant, etc. Therefore in TABLE I, we denote the first two states as $J^{PC}(^3S_1) = 1^{--}$ and the third one as $J^{PC}(^3D_1) = 1^{--}$, etc in turn for high excited states. For the $J^{PC} = 2^{++}$ states, similarly we can see from the figures that the P-wave components f_5 and f_6 are dominant in the first two states (the ground one and the first excited one), and the F-wave components f_3 and f_4 are dominant in the third state (the second excited state), etc. Therefore in TABLE II, we denote the first two states as $J^{PC}(^3P_2) = 2^{++}$ and the third one as $J^{PC}(^3F_2) = 2^{++}$ etc, in turn for high excited states.

Of the $J^{PC} = 1^{--}$ states we may also see from the figure that for the S-wave dominant states they have the properties: $f_3 \simeq -f_4 \equiv \phi_{_{1--}}(q)$ and $f_5 \simeq -f_6 \equiv \psi_{_{1--}}(q)$, so the

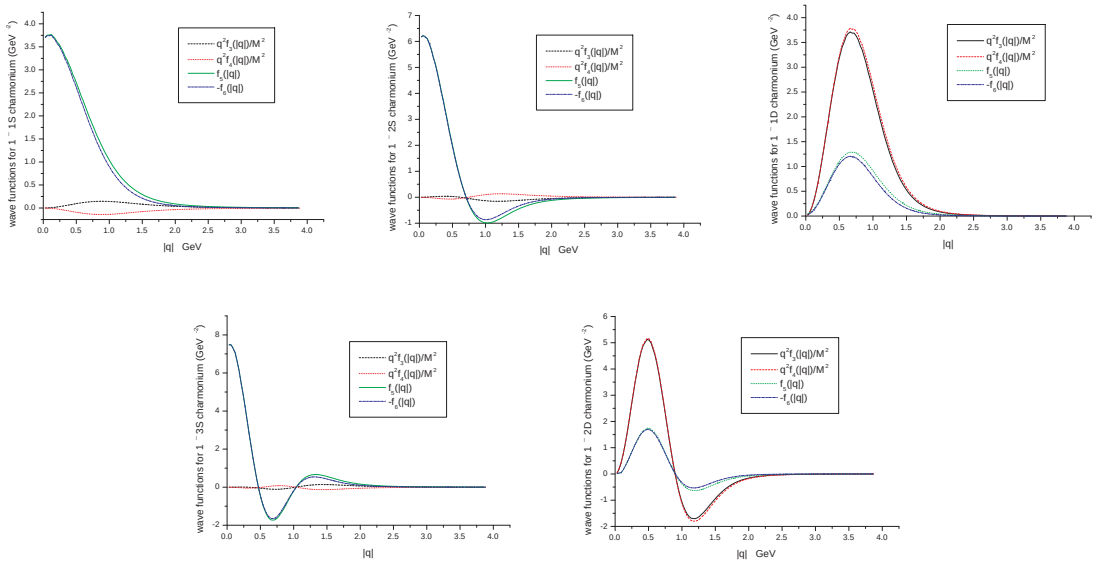

FIG. 5: The wave functions (solutions) of the low-lying states, the ground and the first four excited states, (from left to right) for charmonium with quantum number $J^{PC} = 1^{--}$.

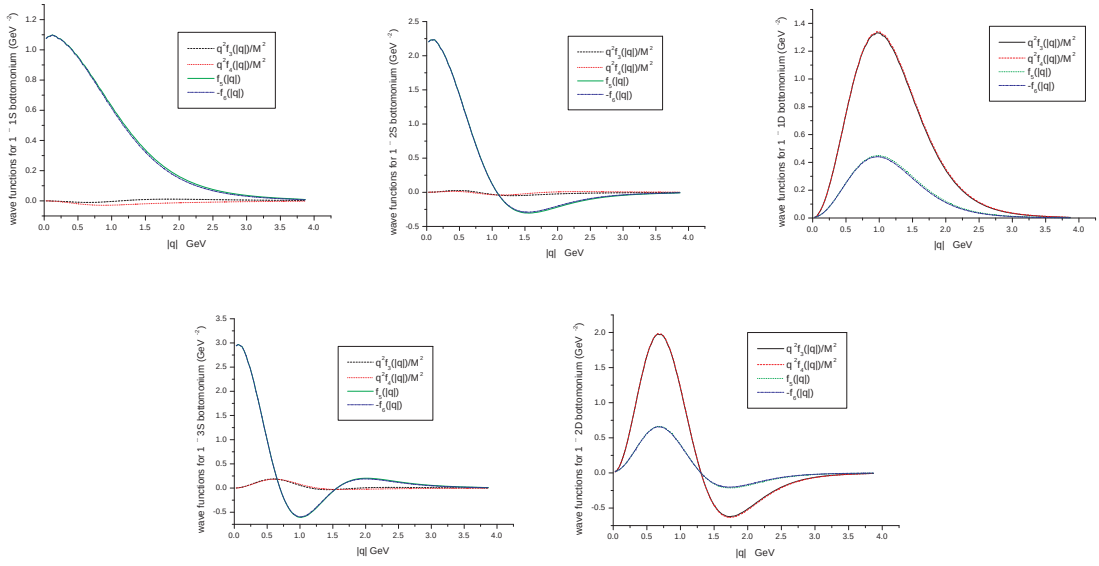

FIG. 6: The wave functions (solutions) of the low-lying states, the ground and the first two excited states, (from left to right) for bottonium with quantum number $J^{PC} = 1^{--}$.

solutions (wave functions) can be re-written (from Eq.(36)) as

$$\varphi^{\lambda}_{P,1^{--}}(q_\perp) \simeq \phi_{1^{--}}(q)(q_\perp \cdot \epsilon^{\lambda}_{\perp})\Big[\Big(\frac{-q^2}{Mm_1} + \frac{\not{q}_\perp}{M}\Big) - \frac{P\not{q}_\perp}{M^2}\Big]$$

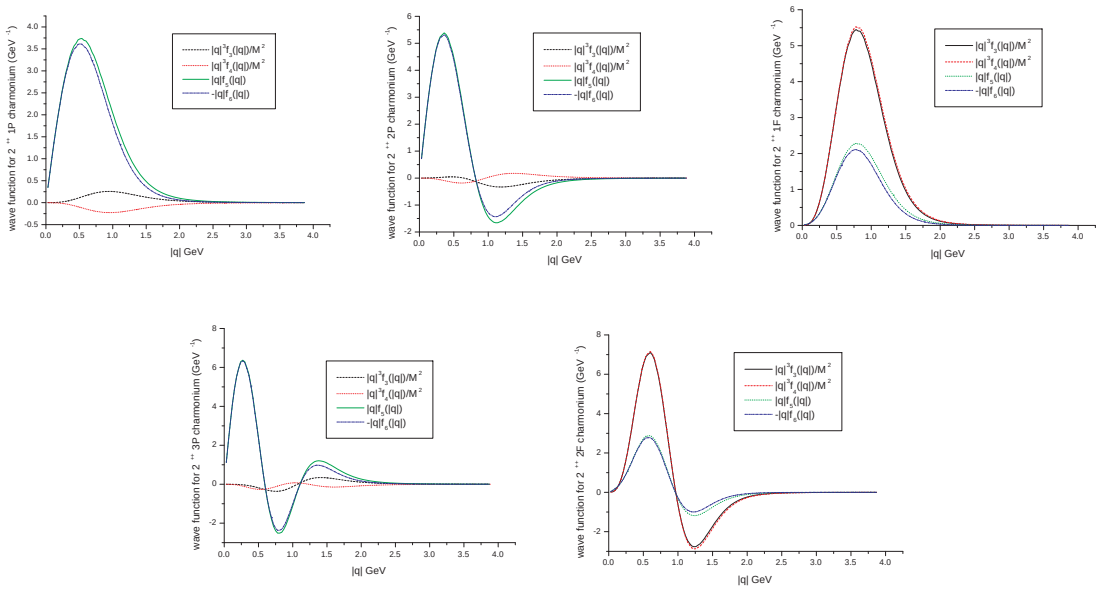

FIG. 7: The wave functions (solutions) of the five low-lying states, the ground and the first four excited states, (from left to right) for charmonium with quantum number $J^{PC} = 2^{++}$.

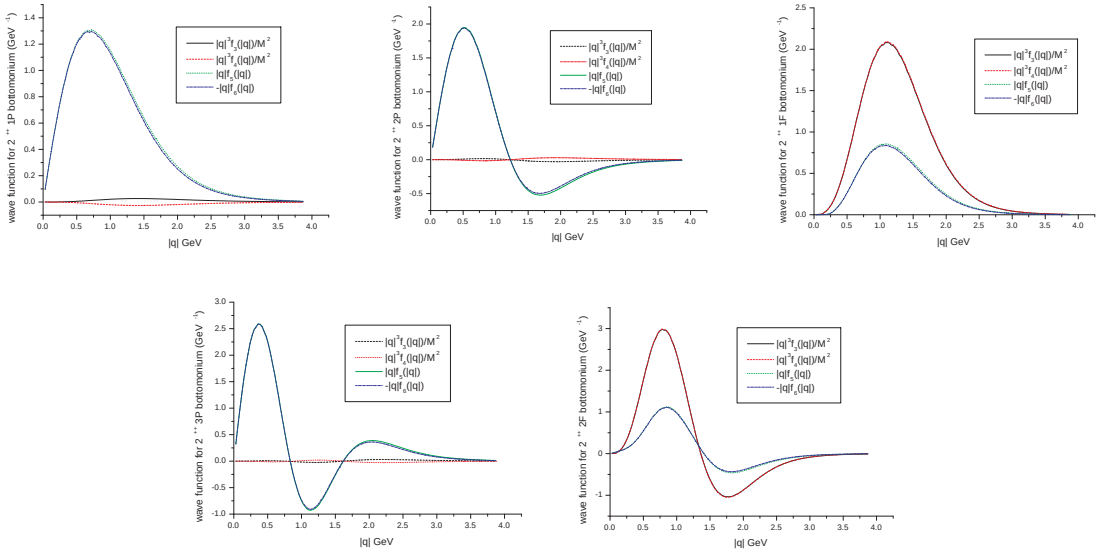

FIG. 8: The wave functions (solutions) of the five low-lying states, the ground and the first four excited states, (from left to right) for bottonium with quantum number $J^{PC} = 2^{++}$.

$$+\ \psi_{_{1--}}(q)\Big\{\Big(M\not{\epsilon}^{\lambda}_{\perp} + q_{\perp}\cdot\epsilon^{\lambda}_{\perp}\frac{M}{m_1}\Big) - \Big[\not{\epsilon}^{\lambda}_{\perp}\not{P} + \frac{\not{P}(q_{\perp}\cdot\epsilon^{\lambda}_{\perp})}{m_1} - \frac{(\not{P}\not{\epsilon}^{\lambda}_{\perp}\not{q}_{\perp})}{m_1}\Big]\Big\}\ ; \quad (55)$$

whereas of the D-wave dominant states they have $f_3 \simeq f_4 \equiv \phi_{_{1--}}(q)$ and $f_5 \simeq -f_6 \equiv$

$\psi_{_{1^{--}}}(q)$, so the solutions (wave functions) can be re-written as

$$\varphi^{\lambda}_{_{P,1^{--}}}(q_\perp) \simeq \phi_{_{1^{--}}}(q)(q_\perp\cdot\epsilon^{\lambda}_\perp)\Big[\Big(\frac{-q^2}{Mm_1}+\frac{\not{q}_\perp}{M}\Big)+\frac{\not{P}\not{q}_\perp}{M^2}\Big] + \psi_{_{1^{--}}}(q)\Big\{\Big(M\not{\epsilon}^{\lambda}_\perp+q_\perp\cdot\epsilon^{\lambda}_\perp\frac{M}{m_1}\Big)-\Big[\not{\epsilon}^{\lambda}_\perp\not{P}+\frac{\not{P}(q_\perp\cdot\epsilon^{\lambda}_\perp)}{m_1}-\frac{(\not{P}\not{\epsilon}^{\lambda}_\perp\not{q}_\perp)}{m_1}\Big]\Big\}\,. \quad (56)$$

From the figures (FIG.7 and FIG.8) of the $J^{PC}=2^{++}$ states, we may see that for the P-wave dominant states, the solutions have the properties: $f_5\simeq -f_6\equiv\psi_{_{2^{++}}}(q)$ and $f_3\simeq -f_4\equiv\phi_{_{2^{++}}}(q)$ and the solutions (wave functions) can be re-written (from Eq.(44)) as

$$\varphi^{\lambda}_{_{P,2^{++}}}(q_\perp)\simeq\phi_{_{2^{++}}}(q)\varepsilon^{\lambda}_{\mu\nu}q^{\nu}_\perp q^{\mu}_\perp\Big[\Big(\frac{\not{q}_\perp}{M}-\frac{q^2}{Mm_1}\Big)-\frac{\not{P}\not{q}_\perp}{M^2}\Big] +\psi_{_{2^{++}}}(q)\varepsilon^{\lambda}_{\mu\nu}q^{\nu}_\perp\Big\{(\gamma^\mu+\frac{q^{\mu}_\perp}{m_1})M-\gamma^\mu\not{P}-i\frac{1}{m_1}\epsilon^{\mu\alpha\beta\gamma}P_\alpha q_{\perp\beta}\gamma_\gamma\gamma_5\Big\}; \quad (57)$$

whereas for the F-wave dominant states, the solutions have the properties: $f_5\simeq f_6\equiv\psi_{_{2^{++}}}(q)$ and $f_3\simeq f_4\equiv\phi_{_{2^{++}}}(q)$ and the solutions (wave functions) can be re-written (from Eq.(44)) as

$$\varphi^{\lambda}_{_{P,2^{++}}}(q_\perp)\simeq\phi_{_{2^{++}}}(q)\varepsilon^{\lambda}_{\mu\nu}q^{\nu}_\perp q^{\mu}_\perp\Big[\Big(\frac{\not{q}_\perp}{M}-\frac{q^2}{Mm_1}\Big)+\frac{\not{P}\not{q}_\perp}{M^2}\Big] +\psi_{_{2^{++}}}(q)\varepsilon^{\lambda}_{\mu\nu}q^{\nu}_\perp\Big\{(\gamma^\mu+\frac{q^{\mu}_\perp}{m_1})M-\gamma^\mu\not{P}-i\frac{1}{m_1}\epsilon^{\mu\alpha\beta\gamma}P_\alpha q_{\perp\beta}\gamma_\gamma\gamma_5\Big\}. \quad (58)$$

Finally we would like to discuss the wave mixture further. As shown in TABLE.II and Eqs.(55,56), each of the states (either charmonium or bottomonium) for $J^{PC}=1^{--}$ contains $S-D$ wave mixing. Some are S-wave dominant and the others are P-wave dominant. The third one of charmonium with $m=3778.9$MeV below the threshold of 'open-charm', for instance, clearly is a D-wave dominant state, and in its decay into $l\bar{l}$, $(l=e,\mu)$ only its S-wave components play a role, so the fraction width of the pure leptonic decay is comparatively small. Indeed it corresponds to the observed one ψ" (with mass $m=3772.92$MeV) as pointed in [2]. Whereas, similarly there is wave mixture for bottomonium too, for instance, once more the third one of bottomonium with $m=10129.5$MeV below the threshold of 'open-bottom' is also a D-wave dominant state and it decays into $l\bar{l},(l=e,\mu)$ only via its S-wave components, so the fractional width of the pure leptonic decay is comparatively small too[3]. Furthermore, the fractional width of the pure leptonic decay for such a state will

[3] In Ref.[2], the wave mixture is obtained via additional interaction and that of charmonium is concerned only, i.e. the mixture for bottomonium is not discussed, that is different from here. Here the mixture for charmonium and bottomonium is fully determined by the B.S. kernel well, so we need to discuss the sense for bottomonium on experimental observations too.

be comparatively much smaller (a quarter) than that of charmonium, due to the two factors that the charge of bottom-quark is smaller than that of charm-quark, and the comparative weight of the S-wave component to the D-wave component in the D-wave dominant state, which is proportional to v^2 ($v_{bottomonium} < v_{charmonium}$) as indicated in Eq.(36)), is small. Therefore such a state is very difficult to be observed either in e^+e^- energy scanning experiments at CLEO and B-factories (due to low production rate) or in hadron colliders (due to very small branching ratio for the lepton pair decay and various backgrounds etc). We believe that all such D-wave dominant states for bottomonium must be still missing in experiments so far, even if our prediction here is true. We conjecture that such states may be observed at Z-factory such as Giga-Z etc elsewhere via $e^+e^- \to (b\bar{b})_{\mathbf{1}^{--}(^3\mathbf{D_1})} + \gamma$ or $e^+e^- \to (b\bar{b})_{\mathbf{1}^{--}(^3\mathbf{D_1})} + \cdots$, because there the backgrounds can be controlled comparatively easy, and numerous such bottomonium states enough for experimental observation can be produced via on-shell Z-boson [20]. For the $P-F$ wave mixture, since the first state with quantum numbers $J^{PC} = 2^{++}$ is a 'high' excited state already so there are only fewer of the $J^{PC} = 2^{++}$ states below the threshold of 'open-charm' or 'open-bottom', thus there are fewer example states which can be used to test the wave mixture, although the tests and the situation essentially are quite similar to the cases of 1^{--} states for $S-D$ wave mixture.

When the bound states does not consist of a pair of quark and antiquark (not as charmonium and bottomonium here), the quantum number C is not a good one, then the present way to solve the problem (the relevant B.S. equation) should be changed accordingly, but its main steps may be still kept and interesting results, which are different from the present, are obtained finally. In fact we have considered the double heavy system $(c\bar{b})$ or $(\bar{c}b)$ as an example for non-(quark-antiquark) binding system and solved the relevant B.S. equation in a similar way, but due to differences we put the results and discussions about the double heavy system $(c\bar{b})$ or $(\bar{c}b)$ elsewhere in Ref.[21].

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